SECONDARY CHARACTERISTIC CLASSES IN K-THEORY

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The object of this paper is to develop a general theory of secondary characteristic classes and to study secondary characteristic classes that arise in K-theory.

Secondary characteristic classes are particularly adapted to studying embedding problems. Massey, and Peterson and Stein developed and exploited secondary characteristic classes in ordinary cohomology theory [11], [17], [18]. On the other hand, Atiyah [3] showed that certain primary characteristic classes γ^i in K-theory give good results for nonembedding of projective spaces. The characteristic classes we develop here were motivated by the desire to define a secondary operation when the top γ^i-class vanishes, in analogy with the operations which arise from the relation $Sq^i(w_n) = w_i \cup w_n$ where w_i is the ith Whitney class of an n-plane bundle.

The viewpoint we will take in this paper is that, in a general cohomology theory, secondary characteristic classes arise in two ways: from a relation between characteristic classes and cohomology operations, or from the degeneration of the Gysin sequence. The organization of the paper is as follows. The first section is preliminary and collects results on spectra, functional operations, representation theory, and K-theory. In §2 we give the various definitions of secondary characteristic classes in the setting of general cohomology theories and principal G-bundles. In §3 we develop the crucial Peterson-Stein formula relating a functional operation in the universal example and the universal secondary characteristic class. We apply this formula in §4 and §5 to study the indeterminacy for the universal secondary characteristic classes and to study the relationship between the various definitions given in §2. In §6 we discuss the secondary characteristic classes in K-theory which arise from the relation $\psi^k(\Delta_{-1}) = \theta_k \Delta_{-1}$, and in §7 and §8 we carry out some computations involving these operations. We should note that throughout this paper H^* will always be a generalized cohomology theory. We hope that a forthcoming paper will deal with the application of these operations to embedding problems.

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1. **Preliminaries.** We will first talk about generalized cohomology theories, and the most complete reference is [22]. P, P_0, and P_2 will be the categories of

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CW-complexes, CW-complexes with base point, and CW-pairs, respectively. On these categories we have the notions of a generalized cohomology theory H^* and a generalized reduced theory $\tilde{H}^*$. We will always assume that our cohomology theory comes from a loop spectrum. If $\{E_n, \varepsilon_n\}$, $\varepsilon_n: S(E_n) \to E_{n+1}$, is such a spectrum, we will always assume that the E_n are spaces having the homotopy type of a CW-complex with base point. We will also assume that $\{E_n, \varepsilon_n\}$ is a ring spectrum [22, p. 254], so that the resulting cohomology theory is multiplicative. We will also have occasion to talk about a stable cohomology operation, which we will take to be a sequence of additive natural transformations $\psi: \tilde{H}^n \to \tilde{H}^{n+N}$ which commute with the boundary map and which are given by maps $h_n: E_n \to E_{n+N}$ with

$$\begin{array}{ccc} S(E_{n-1}) & \longrightarrow & E_n \\ {\scriptstyle S(h_{n-1})}\downarrow & & \downarrow{\scriptstyle h_n} \\ S(E_{n+N-1}) & \longrightarrow & E_{n+N} \end{array}$$

commuting up to homotopy.

(1.1) *Cofibrations.* We will isolate some of the facts we need from [19]. All of the statements in this section will hold in the category of spaces with base point. Let $f: X \to Y$ be a map in this category, and let

$$X \to Y \to C_f \to S(X) \to S(Y) \to \cdots$$

be the corresponding cofibration sequence. Many times we will write $H^n(Y, X)$ for $\tilde{H}^n(C_f) = \Pi(C_f, E_n)$. If we have a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \psi} \\ X' & \xrightarrow[g]{} & Y' \end{array} \tag{1.2}$$

of spaces, then there is an induced map $\chi: C_f \to C_g$ so that in

$$\begin{array}{ccccccccc} X & \xrightarrow{f} & Y & \longrightarrow & C_f & \longrightarrow & S(X) & \longrightarrow & \cdots \\ {\scriptstyle \phi}\downarrow & & {\scriptstyle \psi}\downarrow & & {\scriptstyle \chi}\downarrow & & {\scriptstyle S(\phi)}\downarrow & & \\ X' & \xrightarrow{g} & Y' & \longrightarrow & C_g & \longrightarrow & S(X') & \longrightarrow & \cdots \end{array}$$

all the squares are homotopy commutative. The map χ depends on the homotopy between ψf and $g\phi$; however, if $\psi f = g\phi$, then χ is homotopic to the obvious map from C_f to C_g. If

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ {\scriptstyle \phi'}\downarrow & & \downarrow{\scriptstyle \psi'} \\ X'' & \xrightarrow[h]{} & Y'' \end{array}$$

is another homotopy commutative diagram and if $\chi'\colon C_g \to C_h$ is the induced map, then we can find a homotopy between $\psi'\psi f$ and $h\phi'\phi$ so that the induced map $\chi''\colon C_f \to C_h$ is homotopic to $\chi'\chi$. If both diagrams are actually commutative and if we use the constant homotopy to construct χ and χ' and χ'', then $\chi'\chi$ will be homotopic to χ''. If ϕ_1, ϕ_2 are maps from X to X', ψ_1, ψ_2 are maps from Y to Y' so that $\psi_i f \sim g\phi_i$, $i=1, 2$, and if $\chi_1\colon C_f \to C_g$ is induced from a homotopy between $\psi_1 f$ and $g\phi_1$, then there is some homotopy between $\psi_2 f$ and $g\phi_2$ so that the induced map $\chi_2\colon C_f \to C_g$ is homotopic to χ_1. We should note that if ϕ and ψ are homotopy equivalences in (1.2), so is the induced χ.

Finally, we shall need the fact that if $f\colon X \to Y$ is a map and Z is a space, then

$$Z \wedge X \to Z \wedge Y \to Z \wedge C_f \to Z \wedge S(X) \to \cdots$$

is homotopically equivalent [19, p. 302] to the cofibration sequence for $1 \wedge f$.

(1.3) *Extension of a cohomology theory.* Let S_0 be a category of spaces with base point. We should like to extend our cohomology theory $\tilde{H}^*$ to this category. If X is a space in S_0, $\tilde{H}^n(X)$ will be $\Pi(X, E_n)$. If $f\colon X \to Y$ is a map in S_0, we get a long exact sequence

$$\to H^n(Y, X) \to \tilde{H}^n(Y) \to \tilde{H}^n(X) \to \cdots;$$

and if we have a homotopy commutative diagram of spaces (1.2), we get an induced mapping of long exact sequences once we choose a homotopy between ψf and $g\phi$. We will also be concerned with products. If (Y, X) is an actual pair, we have a product $H^n(Y, X) \otimes H^m(Y) \to H^{n+m}(Y, X)$ which is defined via a diagonal map $Y/X \to Y/X \wedge X$. If $f\colon X \to Y$ is a map in S_0, $u\colon Y \to E_m$, $v\colon C_f \to E_n$ maps representing cohomology classes, we can define the product uv as follows. The commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle f \wedge 1}\downarrow & & \downarrow{\scriptstyle 1 \wedge 1} \\ Y \wedge X & \xrightarrow[1 \wedge f]{} & Y \wedge Y \end{array}$$

gives rise to maps of the corresponding cofibration sequences, and the resulting map $\chi\colon C_f \to Y \wedge C_f$ composed with the map

$$Y \wedge C_f \xrightarrow{u \wedge v} E_m \wedge E_n \longrightarrow E_{m+n}$$

will be the product uv. (We can choose χ so that the maps of the cofibration sequences of f and $1 \wedge f$ give rise to commutative squares.) It is then easy to check that naturality and the boundary formulas [22, 6.19 and 6.20] hold. If Z is a space, $g\colon Z \to Y$ a map, C the mapping cone of $Z \to Y \to C_f$, and $w\colon C \to E_n$ an element of $\tilde{H}(C)$, then we can define the product uw. Namely, the commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & C_f \\ {\scriptstyle g\wedge 1}\downarrow & & \downarrow{\scriptstyle \chi} \\ Y\wedge Z & \longrightarrow & Y\wedge C_f \end{array}$$

gives rise to maps of the corresponding cofibration sequences, and the corresponding $\chi' : C \to Y \wedge C$ can be used to define the product uw.

A convenient category to work with is P_0', the category of spaces having the homotopy type of a CW-complex with base point. From [13] we know that P_0' is closed under products, loop spaces; and if $f: X \to Y$ is a map in P_0', $E \to X$ is the fibering induced from the path-loop fibering of Y, then E is in P_0'.

(1.4) *Functional operations.* Throughout this section ψ will be a stable cohomology operation. Let $f: X \to Y$ be a map in P_0', u an element in $\tilde{H}^n(Y)$, θ in $\tilde{H}^N(Y)$ and suppose that $f^*(u)=0$, $\psi(u)=\theta u$. In this situation we define a functional operation just as in ordinary cohomology [15]. Choose an element v in $H^n(Y, X)$ which restricts to u, then $\psi(u)-\theta u$ restricts to zero in $\tilde{H}^{n+N}(Y)$ and so there is an element x in $\tilde{H}^{n+N-1}(X)$ satisfying $\delta(x)=\psi(v)-\theta v$. The element x is well defined in $\tilde{H}^{n+N-1}(X)$ mod $(\psi-f^*(\theta))\cdot \tilde{H}^{n-1}(X)+f^*\tilde{H}^{n+N-1}(Y)$, and one denotes the coset of x by $[\psi-\theta]_f(u)$. The functional operation satisfies the following properties just as in ordinary cohomology:

(1.5) If $f, g: X \to Y$ are homotopic, then $[\psi-\theta]_f(u)=[\psi-\theta]_g(u)$.

(1.6) If

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \alpha}\uparrow & & \uparrow{\scriptstyle \beta} \\ X' & \xrightarrow[g]{} & Y' \end{array}$$

is homotopy commutative, then

$$[\psi-\beta^*(\theta)]_g(\beta^*(u)) = \alpha^*[\psi-\theta]_f(u) \quad \operatorname{mod} g^*\tilde{H}(Y')+(\psi-\alpha^*(\theta))\tilde{H}(X).$$

(*Note*: By abuse of notation, we sometimes write $[\psi-\theta]_g$ instead of $[\psi-\beta^*(\theta)]_g$.)

(1.7) Let $g: Z \to X$ be a map in P_0', then

$$[\psi-\theta]_{fg}(u) = g^*[\psi-\theta]_f(u) \quad \operatorname{mod} (\psi-g^*f^*(\theta))\tilde{H}(Z)+g^*f^*\tilde{H}(Y).$$

(1.8) Let $f: X \to Y$ and $g: Y \to Z$ be maps in P_0'. Suppose that u is an element in $\tilde{H}^n(Z)$ with $f^*g^*(u)=0$, and θ is an element in $H^N(Z)$ with $\psi(u)=\theta u$. Then

$$[\psi-\theta]_{gf}(u) = [\psi-g^*(\theta)]_f(g^*(u)) \quad \operatorname{mod} f^*\tilde{H}(Y)+(\psi-f^*g^*(\theta))\tilde{H}(X).$$

(1.9) We will need the notion of a functional operation in a slightly more general setting. Let $f: X \to Y$ and $g: Z \to Y$ be maps in P_0'. Let $u \in \tilde{H}^n(C_f)$ and $\theta \in H^N(Y)$. Let $p: Y \to C_f$ and suppose that $g^*p^*(u)=0$ and $\psi(u)=\theta u$. We want to be able to define $[\psi-\theta]_{pg}(u)$ in $\tilde{H}(Z)$. If v is an element in $\tilde{H}^n(C_{pg})$ which restricts to u, then from (1.3) we can define the product θv. Then the standard functional operation construction yields a well-defined element in

$$\tilde{H}(Z) \quad \bmod\ (\psi - g^*(\theta))\tilde{H}(Z) + g^*p^*\tilde{H}(C_f).$$

We denote this coset by $[\psi-\theta]_{pg}(u)$. We will need the following fact, whose proof follows the lines of the standard functional operation argument:

(1.10) LEMMA. *Let $f\colon X \to Y$, $h\colon W \to Z$, $g'\colon W \to X$, $g\colon Z \to Y$ be maps in P_0' such that*

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ h\uparrow & & \uparrow f \\ W & \xrightarrow[g']{} & X \end{array}$$

is commutative. Let $p\colon Y \to C_f$ and $q\colon Z \to C_h$ be the inclusions into the mapping cones, and $g''\colon C_h \to C_f$ the induced map on the mapping cone. Suppose $u \in \tilde{H}^n(C_f)$ and $\theta \in H^N(Y)$ are elements satisfying $p^*g^*(u)=0$ and $\psi(u)=\theta u$. Then

$$[\psi-\theta]_{pg}(u) = [\psi-\theta]_g(p^*(u)) \quad \bmod\ (\psi-g^*(\theta))\tilde{H}(Z)+g^*\tilde{H}(Y)$$

and

$$[\psi-\theta]_{pg}(u) = [\psi-g^*(\theta)]_h(g''^*(u)) \quad \bmod\ (\psi-g^*(\theta))\tilde{H}(Z)+q^*\tilde{H}(C_h).$$

(1.11) *Fundamental class and transgression.* Let $\{E_n, \varepsilon_n\}$ be a loop spectrum so that $\varepsilon_n'\colon E_n \to \Omega E_{n+1}$ is a homotopy equivalence. Then ε_n' induces an isomorphism

$$\Pi(\Omega E_{n+1}, E_n) \to \Pi(E_n, E_n) = \tilde{H}^n(E_n).$$

The fundamental class ι_n will be the class of the identity map in $\tilde{H}^n(E_n)$ or its image in $\tilde{H}(\Omega E_n)$ under the above isomorphism.

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fiber space. Then we have maps $p^*\colon H^n(B, b_0) \to H^n(E, F)$ and $\delta\colon \tilde{H}^{n-1}(F) \to H^n(E, F)$. An element v in $\tilde{H}^{n-1}(F)$ is called transgressive if there is an element x in $H^n(B, b_0)$ so that $p^*(x) = \delta(v)$. In this case we say that v transgresses to x. We have the obvious naturality lemmas:

(1.12) LEMMA. *Let $F \to E \to B$ and $F' \to E' \to B'$ be fiber spaces, and suppose we have a commutative diagram of spaces*

$$\begin{array}{ccc} F' & \xrightarrow{g} & F \\ \downarrow & & \downarrow \\ E' & \xrightarrow{G} & E \\ \downarrow & & \downarrow \\ B' & \xrightarrow{f} & B \end{array}$$

If v in $\tilde{H}^{n-1}(F)$ transgresses to x in $\tilde{H}^n(B)$, then $g^*(v)$ transgresses to $f^*(x)$.

(1.13) LEMMA. *Let $\Omega E_n \to PE_n \to E_n$ be the path-loop fibering of E_n. Then ι_{n-1} in $\tilde{H}^{n-1}(\Omega E_n)$ transgresses to ι_n in $\tilde{H}^n(E_n)$.*

(1.14) *Facts about K-theory.* We will gather facts about K-theory that we need in our computations. The main references are [1], [4], [5], [7]. We will use the notation KU and KO for complex and real K-theories, and we will use K when we are making a statement about both. $RU(G)$ and $RO(G)$ will be the complex and real representation rings of the compact connected Lie group G, and $RSp(G)$ will be the symplectic representation group. KU-theory comes from the loop-spectrum of BU, $E_n = \Omega^n BU$, $n \leqq 0$, and E_n is defined for positive n by using periodicity $BU \to \Omega^2 BU$. KO-theory comes from the loop spectrum of BO, $E_n = \Omega^n BO$ for $n \leqq 0$, and E_n is defined for positive n by periodicity. We also have KSp-theory which comes from BSp and natural transformations ε_U: $KO \to KU$, ε_{Sp}: $KU, KO \to KSp$.

If X is a finite complex, both $KU^*(X)$ and $KO^*(X)$ have ring structures due to the tensor product. The following theorem due to Donald Anderson ([1] and secret communication) relates this ring structure to a ring structure on spectra.

(1.15) THEOREM (ANDERSON). *The spectra of KU and KO theory both have unique ring structures which restricts to the ordinary tensor product for finite complexes.*

(1.16) *Adams operations.* By exactly the same techniques as the proof of (1.15) one can show that Adams operations ψ^k: $K^i(X) \to K^i(X)$ extend uniquely to maps of the spectra h_n: $E_n \to E_n$. This gives rise to a stable operation in negative dimensions (since δ is given by a map of spaces) but ψ^k does not commute with periodicity so that ψ^k is not stable in positive dimensions. We will only be concerned with the stability of ψ^k from dimension zero to dimension -1.

From (1.1) and (1.3) we can think of KU^* and KO^* as multiplicative theories with stable operation ψ^k as a functor on the category of CW pairs or on the category of spaces having the homotopy type of a CW-complex.

(1.17) *Thom isomorphism* [5], [7]. If P is a principal spin $(2n)$ bundle over a finite complex X, E the associated complex vector bundle, then there is an element U in $KU^0(D(E), S(E))$ so that

$$\otimes\, U\colon KU^*(X) \to KU^*(D(E), S(E))$$

is an isomorphism. Further $\psi^k(U) = \theta_k(P)U$, and the restriction of U to $KU^*(X)$ is $\Delta_+(P) - \Delta_-(P) = \Delta_{-1}(P)$. Of course, then it follows that $\psi^k(\Delta_{-1}(P)) = \theta_k(P)\Delta_{-1}(P)$. Here $\Delta_+, \Delta_-, \theta_k$ are complex virtual representations (see [7, p. 64]).

If P is a real spin $(8n)$-bundle over X, and E is its associated real vector bundle, there is an element U in $KO^0(D(E), S(E))$ so that

$$\otimes\, U\colon KO^*(X) \to KO^*(D(E), S(E))$$

is an isomorphism. Again $\psi^k(U) = \theta_k(P)U$, U restricts to $\Delta_+(P) - \Delta_-(P) = \Delta_{-1}(P)$, and $\psi^k(\Delta_{-1}(P)) = \theta_k(P)\Delta_{-1}(P)$. Here $\Delta_+, \Delta_-, \theta_k$ are real virtual representations.

(1.18) $RU(U(n))$ [7], [14]. We let $U(n)$ be the $n\times n$ unitary matrices and T the maximal torus consisting of all diagonal matrices. The Weyl group is generated by permutations of diagonal elements. If $t=\operatorname{diag}(\exp(i\theta_1),\ldots,\exp(i\theta_n))$ and $\alpha_j(t)=\exp(i\theta_j)$, then

$$RU(U(n)) = Z[\alpha_1,\ldots,\alpha_n,\alpha_1^{-1},\ldots,\alpha_n^{-1}].$$

Atiyah [3] has introduced virtual representations γ^i which are best described by their characters

$$\gamma_t = 1+\gamma^1 t+\cdots+\gamma^n t^n = \prod_n (1+t(\alpha_i-1))$$

$Z[\gamma^1,\ldots,\gamma^n]$ is a subring of $RU(U(n))$. If we complete $Z[\alpha_1-1,\ldots,\alpha_n-1]$ with respect to $(\alpha_1-1,\ldots,\alpha_n-1)$, then from [6, p. 30] it follows that the injection into $RU(T)$ induces a topological isomorphism $Z[[\alpha_1-1,\ldots,\alpha_n-1]]\to \hat{R}U(T)$ and this implies

$$\lim\operatorname{inv} KU(BU(n)) = \hat{R}U(U(n)) = Z[[\gamma^1,\ldots,\gamma^n]].$$

We should note that upon restriction from $U(n)$ to $U(n-1)$, $\gamma^1,\ldots,\gamma^{n-1}$ restrict to $\gamma^1,\ldots,\gamma^{n-1}$ and γ^n restricts to zero.

(1.19) *Representations of* $Sp(n)$. Let Q^n be quaternionic n-space and let ρ be the standard representation of $Sp(n)$ on Q^n. If we ignore the quaternionic structure on Q^n, so that we think of $Sp(n)\subset U(2n)$, then ρ is the standard representation of $U(2n)$ on C^{2n}. Let $T\subset Sp(n)\subset U(2n)$ consist of the elements

$$t = \operatorname{diag}(\exp(i\theta_1),\exp(-i\theta_1),\ldots,\exp(i\theta_n),\exp(-i\theta_n)).$$

T is a maximal torus in $Sp(n)$. If α_j is the character $\alpha_j(t)=\exp(i\theta_j)$, then

$$RU(T) = Z[\alpha_1,\ldots,\alpha_n,\alpha_1^{-1},\ldots,\alpha_n^{-1}].$$

The Weyl group is generated by the permutations $\alpha_i\to\alpha_j$ and $\alpha_i\to\alpha_i^{-1}$ [20, Exposé 1]. The exterior powers of the representation ρ are given by

$$\lambda_t = \prod_{i=1}^{n} (1+\alpha_i t)(1+\alpha_i^{-1}t)$$

and a computation similar to the one in [14] for $SO(n)$ shows that

$$RU(Sp(n)) = Z[\lambda^1,\ldots,\lambda^n].$$

Anderson ([2] and clandestine communication) has introduced certain virtual representations π^i. These are best described by their characters

$$\pi_t = 1+\pi^1 t+\cdots+\pi^n t^n = \prod_{i=1}^{n} (1+(\alpha_i+\alpha_i^{-1}-2)t).$$

It follows easily that $\lambda_t=(1+t)^{2n}\pi_{t/(1+t)^2}$ so that

$$RU(Sp(n)) = Z[\pi^1,\ldots,\pi^n].$$

We want to know which of the π^i are real and which are symplectic. The following lemma follows easily from the fact that a self-conjugate representation of a compact Lie group is real or symplectic depending upon whether it admits an invariant symmetric nondegenerate form or an invariant antisymmetric nondegenerate form.

(1.20) LEMMA. *Let G be a compact Lie group and ρ a complex representation which is symplectic. Then $\lambda^i(\rho)$ is real if i is even and $\lambda^i(\rho)$ is symplectic if i is odd.*

(1.21) LEMMA. *π^{2i} is real and π^{2i+1} is symplectic.*

Proof. From $\lambda_t = \pi_{t/(1+t^2)}$ it follows that

$$\lambda^m = \binom{2n}{m} + \binom{2n-2}{m-1}\pi^1 + \binom{2n-4}{m-2}\pi^2 + \cdots + \pi^m.$$

Now the lemma follows by induction and from the facts that $C_{2r,s}$ is even if s is odd, and that twice a symplectic representation is both real and symplectic.

We should remark that under restriction from $S_p(n)$ to $S_p(n-1)$, $\pi^1, \ldots, \pi^{n-1}$ restrict to $\pi^1, \ldots, \pi^{n-1}$ and π^n restricts to zero.

(1.22) *Difference element of two representations.* Let G be a compact connected Lie group, H a closed subgroup and F will be either $U(n)$, $O(n)$, or $S_p(n)$. Suppose α and β are two F-representations which become equivalent when restricted to H. Let σ be an element of F such that $\alpha(h) = \sigma\beta(h)\sigma^{-1}$ for each h in H. The map $g \to \alpha(g)\sigma\beta(g^{-1})\sigma^{-1}$ passes to a map of $G/H \to F$ and so gives rise to an element in $\tilde{K}U^{-1}(G/H)$, $\tilde{K}O^{-1}(G/H)$, or $\tilde{K}Sp^{-1}(G/H)$. We will denote this element (α, β, σ). If $S \subset H$ is a closed subgroup, then α, β, and σ give rise via this construction to an element in the appropriate K-group of G/S and via the restriction $K(G/H) \to K(G/S)$, the two difference elements correspond to one another.

(1.23) *KU-theory of complex Stiefel manifolds.* We know from (1.18) that the virtual representation γ^i of $U(n)$ restricts to zero on $U(i-1)$. We can then apply the construction of (1.22) to get elements in $\tilde{K}U^{-1}(U(n)/U(m))$ for $m \leqq i-1$ which we also call γ^i. In particular, from [4, p. 115], we can choose γ^n to be the generator of $\tilde{K}U^{-1}(S_{2n-1})$. If we let g be the generator of $KU^{-2}(\text{point})$, so that $KU^* = KU^*(\text{point}) = Z[g]$, then $KU^*(S_{2n-1})$ is freely generated by γ^n over KU^*. Let $\Lambda\,[x_1, \ldots, x_r]$ denote the exterior algebra generated by $x_1, \ldots, x_r$.

(1.24) LEMMA. $KU^*(U(n)/U(m)) = \Lambda[\gamma^{m+1}, \ldots, \gamma^n]$.

Proof. We use Dold's theorem [8]. We apply this theorem to $U(m) \to U(n) \to U(n)/U(m)$. By induction (or by [4, p. 116]) we know that $KU^*(U(n)) = \Lambda[\gamma^1, \ldots, \gamma^n] \otimes KU^*$. The elements $\gamma^1, \ldots, \gamma^m$ restrict to generators of $KU^*(U(m))$ so that $KU^*(U(n)/U(m))$ is the subalgebra generated by $\gamma^{m+1}, \ldots, \gamma^n$.

(1.25) *Symplectic Stiefel manifolds.* From (1.19) we know that the virtual representations π^i of $Sp(n)$ restrict to zero on $Sp(i-1)$ so that in $\tilde{K}U^{-1}(Sp(n)/Sp(m))$ we have elements $\pi^{m+1}, \ldots, \pi^n$ which restrict to one another. On

$$S_{4n-1} = U(2n)/U(2n-1) = Sp(2n)/Sp(2n-1)$$

we have $\gamma^{2n}=(-1)^n\pi^n$ so that π^n is a generator of $\tilde{K}U^{-1}(S_{4n-1})$. Just as in (1.23) we have

(1.26) LEMMA. $KU^*(Sp(n)/Sp(m))=\Lambda[\pi^{m+1},\ldots,\pi^n]\otimes KU^*$.

The ring $KO^*=KO^*(\text{point})$ is generated by $\xi\in KO^{-1}$, $\mu\in KO^{-4}$, $\eta\in KO^{-8}$ subject to $2\xi=0$, $\xi^3=0$, $\mu^2=4\eta$. The map $\varepsilon_U\colon KO^*\to KU^*$ takes ξ to zero, μ to $2g^2$, and η to g^4.

The virtual representations π^i for i even are real, restrict to zero in $RO(Sp(i-1))$ and so give rise to elements π^i in $\tilde{K}O^{-1}(Sp(n)/Sp(m))$ for $m\leqq i-1$ which restrict to one another. For i odd, π^i gives rise to elements π^i in $\tilde{K}Sp^{-1}(Sp(n)/Sp(m))$ $=\tilde{K}O^{-5}(Sp(n)/Sp(m))$, $m\leqq i-1$, which restrict to one another. The map

$$\otimes g^{-2}\cdot\varepsilon_U\colon KO^{-5}(Sp(n)/Sp(m))\to KO^{-1}(Sp(n)/Sp(m))$$

takes π^{2i+1} to π^{2i+1}. Now, just as in (1.23) and (1.25) we have

(1.27) LEMMA. $KO^*(Sp(n)/Sp(m))=\Lambda[\pi^{2i},\pi^{2i+1}]\otimes KO^*$ *for* $m+1\leqq 2i$, $2i+1\leqq n$.

2. **Definition of secondary characteristic classes.** Our purpose here is to make two constructions. First (2.1) corresponding to a relation among characteristic classes and cohomology operations, we will construct a universal fiber space and universal cohomology classes which will eventually represent a secondary operation Φ. Second (2.3) given corresponding data about the Thom isomorphism, we will construct another secondary operation Ψ when the Gysin sequence degenerates. Both methods are familiar in ordinary cohomology [11], [18].

(2.1) We will assume we are given a cohomology theory H^* which comes from a loop spectrum $\{E_n\}$ with ring structure and stable operation ψ as in 1. Let G be a compact Lie group and let β be the category consisting of pairs (P, X) where X is a finite CW-complex with base point, and $\pi\colon P\to X$ is a principal G-bundle. A map $(P, X)\to(P', X')$ will be a pair of maps $f\colon X\to X'$ base point preserving, and $g\colon P\to P'$ a map of principal G-bundles satisfying $\pi'g=f\pi$. We will suppose that we are given a natural transformation α from β to $\tilde{H}^n$ and a natural transformation θ from β to H^N so that if P is a principal G-bundle over the finite complex X we have the relation

$$\psi(\alpha(P)) = \theta(P)\alpha(P) \tag{2.2}$$

in $\tilde{H}^{n+N}(X)$. For technical convenience, we will assume that either N is even or $+1=-1$ in $H^*(X)$.

The situation is quite reasonable. If $\beta=O(n)$ bundles, $\alpha=w_n=n$th Whitney class, $\psi=Sq^k$, $\theta=w_k=k$th Whitney class, and $H^*=$ ordinary Z_2 cohomology, we have the relation $Sq^k(w_n)=w_k\cup w_n$. This is essentially the situation in [18]. We shall be more interested in $\beta=\text{spin}\,(2n)$ or spin $(8n)$-bundles, $H^*=KU^*$ or KO^*, $\alpha=\Delta_{-1}$, $\theta=\theta_k$, $\psi=\psi^k$ and from (1.17) $\psi^k(\Delta_{-1}(P))=\theta_k(P)\Delta_{-1}(P)$ in $KU^*(X)$ or $KO^*(X)$.

We let B be a classifying space for G-bundles over complexes of dimension less than a fixed number, so we can take B to be a finite complex. Corresponding to the universal G-bundle over B we have universal cohomology classes α in $\tilde{H}^n(B)$ and θ in $H^N(B)$. We let θ_0 be the restriction of θ to a point. Let $f: B \to E_n$ represent α, so that $f^*(\iota_n) = \alpha$. Let $\pi: E \to B$ be the fiber space over B with fiber ΩE_n induced over B from the path loop fibering of E_n. Thus $\pi^*(\alpha) = 0$ in $\tilde{H}^n(E)$. From (1.12) and (1.13) ι_{n-1} transgresses to α in $\Omega E_n \to E \to B$. In the sequence

$$\tilde{H}(\Omega E_n) \xrightarrow{\delta} H(E, \Omega E_n) \longleftarrow H(B, b_0)$$

we have $\delta(\psi(\iota_{n-1}) - \theta_0 \iota_{n-1}) = \pi^*(\psi(\alpha) - \theta\alpha) = 0$. Thus there are elements ξ in $\tilde{H}^{n+N-1}(E)$ which restrict to $\psi(\iota_{n-1}) - \theta_0 \iota_{n-1}$. We will want some such ξ to represent a universal cohomology operation. However if $h: X \to B$ represents a principal G-bundle over X, and $h': X \to E$ is a lifting of h, $\pi h' \sim h$, the homotopy class of h' is not determined by that of h, so there will be an indeterminacy. We will turn to this problem in §4.

(2.3) *Thom space construction.* Let ρ be a real or complex representation of G. Let P be a principal G-bundle over X, $E = \rho(P)$ the corresponding real or complex vector bundle. We will suppose that P has a Thom class U in $H^*(D(E), S(E))$ so that

$$\otimes U: H^*(X) \to H^*(D(E), S(E))$$

is an isomorphism and U satisfies the following conditions:

1. U is natural with respect to maps of G-bundles.
2. $\psi(U) = \theta(P)U$.
3. If $i: X \to D(E)/S(E)$ is the inclusion of X as the zero section followed by projection onto the quotient, then $i^*(U) = \alpha(P)$.

This is the situation if ρ is the identity map of $O(n)$, $\psi = Sq^k$, $\theta = w_k$, $\alpha = w_n$ and $H^* = Z_2$ cohomology. These properties also hold if we take ρ to be the standard representation of spin $(2n)$ or spin $(8n)$ on C^{2n} or R^{8n}, $\psi = \psi^k$, $\alpha = \Delta_{-1}$, $\theta = \theta_k$, and $H^* = KU^*$ or KO^*.

Now suppose P is a bundle for which $\alpha(P) = 0$, then the long exact sequence for $(D(E), S(E))$ breaks up into short exact sequences

$$0 \to \tilde{H}^i(X) \to \tilde{H}^i(S(E)) \xrightarrow{\delta} H^{i+1}(D(E), S(E)) \to 0.$$

Let a be an element in $\tilde{H}^{n-1}(S(E))$ such that $\delta(a) = U$. Then every element in $\tilde{H}^i(S(E))$ can be written uniquely as $xa + y$ where x is in $\tilde{H}^{i-(n-1)}(X)$ and y is in $\tilde{H}^i(X)$. In particular, write $\psi(a) = xa + y$. Then we apply δ and find that $x = \theta(P)$. If a' is another element with $\delta(a') = U$, then $\psi(a') = \theta(P)a' + y'$. Then $y - y'$ lies in $(\psi - \theta(P))\tilde{H}^{n-1}(X)$. Thus we can define a natural transformation Ψ, from principal G-bundles whose α-class vanishes, to a natural quotient of H^*. If $P \to X$ is such a bundle, $\Psi(P, X)$ takes values in $\tilde{H}^*(X) \bmod (\psi - \theta(P))\tilde{H}^*(X)$ and is the coset of y.

3. **The Peterson-Stein formula.** We will prove theorems which will allow us to compute the indeterminacy of the operation we introduced in (2.1) and relate this operation to the one introduced in (2.3). These theorems are generalizations of certain theorems in [17] for ordinary cohomology. We aim at proving

(3.1) THEOREM. *Let* $F \xrightarrow{k} E \xrightarrow{q} X$ *be a fiber space. Suppose* v *in* $\tilde{H}^{n-1}(F)$ *transgresses to* x *in* $\tilde{H}^n(B)$, *and* θ *in* $H^N(B)$ *is such that* $\psi(x)=\theta x$. *Then the functional operation* $[\psi-\theta]_q(x)$ *is defined and*

$$k^*[\psi-\theta]_q(x) = -(\psi(v)-\theta_0 v) \mod (\psi-\theta_0)k^*\tilde{H}^{n+N-1}(E).$$

(*Note.* As before, θ_0 is the restriction of θ to a point.)

Referring to the notation of (2.1), we will use Theorem (3.1) to prove

(3.2) THEOREM. *In the fiber space* $\Omega E_n \xrightarrow{i} E \xrightarrow{\pi} B$, *we can choose an element* ξ_0 *in* $\tilde{H}^{n+N-1}(E)$ *such that*

1. ξ_0 *restricts to* $\psi(\iota_{n-1})-\theta_0\iota_{n-1}$ *in* $\tilde{H}^{n+N-1}(\Omega E_n)$.
2. $-\xi_0$ *represents* $[\psi-\theta]_\pi(\alpha)$.

We shall devote the remainder of this section to proving these two theorems.

(3.3) LEMMA. *Let* X, Y, Z *be spaces having the homotopy type of* CW-*complexes with base point. Let* $f\colon X\times Y\to Z$ *be a base point preserving map, and* $\Omega Z\to B \to X\times Y$ *the fiber space induced from the path loop fibering of* Z. *Let* $B\to X$ *be projection onto the first factor, and* $\Omega X \xrightarrow{j} C\to B$ *the fiber space induced from the path loop fibering of* X. *Then there is a map* $h\colon \Omega Z\to C$ *so that*

$$\begin{array}{ccc} \Omega X\times\Omega Y & \xrightarrow{t} & \Omega X \\ {\scriptstyle f^1}\downarrow & & \downarrow{\scriptstyle j} \\ \Omega Z & \xrightarrow{h} & C \end{array}$$

is homotopy commutative, where $f^1=\Omega f$ *and* t *is the projection onto the first factor followed by taking the inverse of a loop.*

Proof. The space B consists of all triples (x, y, ω) where ω is a path in Z and $f(x, y)=\omega(1)$. The space C consists of all (x, y, ω, λ) where ω is a path in Z, λ a path in X, and $f(x, y)=\omega(1)$, $\lambda(1)=x$. Now let γ be in ΩZ. We define h by

$$h(\gamma) = (*, *, \gamma, c_0), \tag{3.4}$$

where $*$ is the base point, and c_0 is the constant path at the base point. Then $hf^1(\alpha, \beta)=(*, *, f(\alpha(t), \beta(t)), c_0)$. We define a homotopy $F\colon \Omega X\times\Omega Y\times[0, 1]\to C$ by

$$F(\alpha, \beta, s) = (*, *, f(\alpha(t), \beta(t)), \sigma_s(t))$$

where $\sigma_s(t)=\alpha(1-2t(1-s))$ for $0\leqq t\leqq \frac{1}{2}$ and $\sigma_s(t)=\alpha((1-s)(2t-1)+s)$ for $\frac{1}{2}\leqq t\leqq 1$. Then we see that $F_0(\alpha, \beta)=(*, *, f(\alpha, \beta), \alpha\cdot\alpha^{-1})$, $F_1=h\Omega f$. Now we define

$G\colon \Omega X \times \Omega Y \times [0, 1] \to C$ by

$$G(\alpha, \beta, s) = (\alpha(1-s), \beta(1-s), f(\alpha(1-s)t), \beta((1-s)t), \tau_s(t))$$

where $\tau_s(t) = \alpha(1-2t)$ for $0 \leqq t \leqq \frac{1}{2}$ and $\tau_s(t) = \alpha((2t-1)(1-s))$ for $\frac{1}{2} \leqq t \leqq 1$. We easily check that $G_0 = F_0$ and $G_1(\alpha, \beta) = (*, *, c_0, c_0 \cdot \alpha^{-1})$. Then it is clear that G_1 is homotopic to jt.

We will apply this lemma in the following situation. We have ι_n in $\tilde{H}^n(E_n)$, ι_N in $\tilde{H}^N(E_N)$, and $\psi(\iota_n) - (\iota_n \oplus \theta_0)\iota_n$ in $\tilde{H}^{n+N}(E_n \times E_N)$. We represent this element by a map $f\colon E_n \times E_n \to E_{n+N}$. The map $h\colon \Omega E_{n+N} \to C$ given by $h(\gamma) = (*, *, \gamma, c_0)$ makes

$$\begin{array}{ccc} \Omega E_n \times \Omega E_n & \xrightarrow{t} & \Omega E_n \\ {\scriptstyle f^1}\downarrow & & \downarrow{\scriptstyle j} \\ \Omega E_{n+N} & \xrightarrow{h} & C \end{array}$$

homotopy commutative.

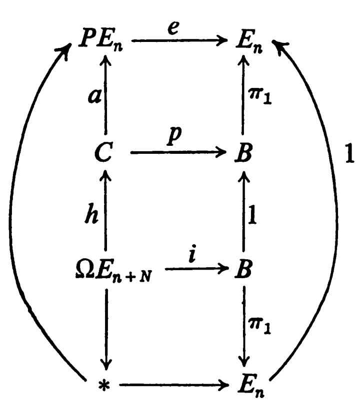
We have a commutative diagram

$$\begin{array}{ccc} \Omega E_{n+N} & \xrightarrow{h} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[1]{} & B \end{array}$$

so we have the induced map $\chi\colon (B, \Omega E_{n+N}) \to (B, C)$, as well as the projection $\sigma\colon (B, \Omega E_{n+N}) \to (E_n \times E_N, *)$. We can think of ι_n as an element in $H^n(E_n \times E_N, *)$.

(3.4) Lemma. *We can choose an element v in $H^n(B, C)$ so that $\chi^*(v) = \sigma^*(\iota_n)$.*

Proof. We have a big commutative diagram



where $e\colon PE_n \to E_n$ is the projection in the path loop fibering, $a(x, y, \omega, \gamma) = \gamma$, $\pi_1(x, y, \omega) = x$, $e(\gamma) = \gamma(1)$. We apply (1.1) to get a commutative diagram of maps on the mapping cones and so a commutative diagram

$$\begin{array}{ccccc} H(E_n, PE_n) & \xrightarrow{\cong} & H(E_n, *) & \longrightarrow & H(B, \Omega E_{n+N}) \\ & \searrow & & \nearrow{\scriptstyle \chi^*} & \\ & & H(B, C) & & \end{array}$$

We can start with ι_n in $H(E_n, PE_n)$ and its image in $H(B, C)$ will be the desired v.

Now we repeatedly apply (1.12) and (1.13) to obtain the following formulas:

(3.5) In $\Omega E_{n+N} \xrightarrow{i} B \rightarrow E_n \times E_N$, ι_{n+N-1} transgresses to $\psi(\iota_n) - (\iota_N \oplus \theta_0)\iota_n$ so that in $H^{n+N}(B)$ we have $\psi(\iota_n) - (\iota_N \oplus \theta_0)\iota_n = 0$.

(3.6) In $\Omega E_n \xrightarrow{j} C \xrightarrow{p} B$, ι_{n-1} transgresses to ι_n in $\tilde{H}^n(B)$ so that $p^*(\iota_n) = 0$. Combining this with (3.5) we see that $[\psi - (\iota_N \oplus \theta_0)]_p(\iota_n)$ is defined.

(3.7) By applying the path-loop functor to the map $f\colon E_n \times E_N \rightarrow E_{n+N}$ we find that, in the fiber space

$$\Omega E_n \times \Omega E_N \rightarrow PE_n \times PE_N \rightarrow E_n \times E_N,$$

$f^{1*}(\iota_{n+N-1})$ transgresses to $\psi(\iota_n) - (\iota_N \oplus \theta_0)\iota_n = \psi(\iota_n) - \theta_0\iota_n$.

(3.8) By projecting onto the first factor in

$$\Omega E_n \times \Omega E_N \rightarrow PE_n \times PE_N \rightarrow E_n \times E_N$$

we see that ι_{n-1} transgresses to ι_n and so $\psi(\iota_{n-1}) - \theta_0\iota_{n-1}$ transgresses to $\psi(\iota_n) - \theta_0\iota_n$.

Now we are ready to prove the universal form of Theorem (3.1).

(3.9) THEOREM. *In* $\Omega E_n \xrightarrow{j} C \xrightarrow{p} B$

$$j^*[\psi - (\iota_N \oplus \theta_0)]_p(\iota_n) \;=\; -(\psi(\iota_{n-1}) - \theta_0\iota_{n-1}) \quad \text{mod } (\psi - \theta_0)j^*\tilde{H}^{n-1}(C).$$

Proof. From Lemma (3.4) the element v restricts to ι_n in $\tilde{H}(B)$, and so by (3.5) $\psi(v) - (\iota_N \oplus \theta_0)v$ restricts to zero in $\tilde{H}(B)$. Thus there is a u in $\tilde{H}^{n+N-1}(C)$ such that

$$\delta(u) \;=\; \psi(v) - (\iota_N \oplus \theta_0)v. \tag{3.10}$$

We have the map $(B, C) \rightarrow (B, \Omega E_{n+N})$ and from (3.10) and naturality we get

$$\delta h^*(u) \;=\; \psi(\iota_n) - (\iota_N \oplus \theta_0)\iota_n \tag{3.11}$$

in $H(B, \Omega E_{n+N})$. (Strictly speaking, we should write $\sigma^*(\iota_n)$ instead of ι_n in concordance with (3.4).) Comparing (3.11) with (3.5) we see that $h^*(u) - \iota_{n+N-1} = i^*(x)$ for some x in $\tilde{H}^{n+N-1}(B)$. Now apply f^{1*} to this relation. Since if^1 is homotopic to a constant, $f^{1*}h^*(u) = f^{1*}(\iota_{n+N-1})$. By (3.7) $f^{1*}(\iota_{n+N-1})$ transgresses to $\psi(\iota_n) - \theta_0\iota_n$, and by (3.8) so does $\psi(\iota_{n-1}) - \theta_0\iota_{n-1}$; hence $f^{1*}h^*(u) = \psi(\iota_{n-1}) - \theta_0\iota_{n-1}$ in $\tilde{H}(\Omega E_n \times \Omega E_N)$.

Now we compare this with (3.3) and find

(3.12) $$t^*j^*(u) = \psi(\iota_{n-1}) - \theta_0\iota_{n-1}.$$

Let $\eta\colon \Omega E_N \to \Omega E_n \times \Omega E_N$ take a loop ω to (ω^{-1}, c_0) so that $t\eta = 1$. Thus from (3.12) we get

(3.13) $$j^*(u) = \psi(\eta^*(\iota_{n-1}) - \theta_0\eta^*(\iota_{n-1})) = -(\psi(\iota_{n-1}) - \theta_0\iota_{n-1}).$$

Now, from (3.10), u represents $[\psi - (\iota_N \oplus \theta_0)]_p(\iota_n)$ with indeterminacy $(\psi - (\iota_N \oplus \theta_0))\tilde{H}(C) + p^*\tilde{H}(B)$. If we compare this with (3.13) we get the desired result.

Proof of Theorem (3.1). We retain the notation of the previous theorem. Let $h_1\colon X \to E_n$ represent x and let $h_2\colon X \to E_N$ represent $\theta - \theta_0$. Then the composite $f(h_1 \times h_2)\colon X \to E_n \times E_N$ is homotopically trivial so that $h_1 \times h_2$ lifts to a map $h_0\colon X \to B$. Since $h_0^*(\iota_n) = x$ and $q^*(x) = 0$, the map $h_0 q$ lifts to a map $h'\colon E \to C$ and we can choose h' so that $ph' = h_0 q$. Thus $h'' = (h' \mid F)\colon F \to \Omega E_n$. By (1.7) and (1.8) we have

$$[\psi - \theta]_q(x) = h'^*[\psi - (\iota_N \oplus \theta_0)]_p(\iota_n) \mod q^*\tilde{H}(X) + (\psi - \theta)\tilde{H}(E).$$

From (3.9) we get

$$k^*[\psi - \theta]_q(x) = -h''^*(\psi(\iota_{n-1}) - \theta_0\iota_{n-1}) \mod (\psi - \theta_0)k^*\tilde{H}(E).$$

Now, by naturality, $h''^*(\iota_{n-1})$ transgresses to x, so that $v - h''^*(\iota_{n-1}) = k^*(y)$. Thus

$$\psi(v) - \theta_0 v = h''^*(\psi(\iota_{n-1}) - \theta_0\iota_{n-1}) \mod (\psi - \theta_0)k^*\tilde{H}(E)$$

and so

$$k^*[\psi - \theta]_q(x) = -(\psi(v) - \theta_0 v) \mod (\psi - \theta_0)k^*\tilde{H}(E).$$

Proof of Theorem (3.2). Referring to (2.1), we have our universal example $\Omega E_n \xrightarrow{i} E \xrightarrow{\pi} B$ and elements in $\tilde{H}^{n+N-1}(E)$ which restrict to $\psi(\iota_{n-1}) - \theta_0\iota_{n-1}$. From our construction ι_{n-1} transgresses to α, and so by (3.1)

$$i^*[\psi - \theta]_\pi(\alpha) = -(\psi(\iota_{n-1}) - \theta_0\iota_{n-1}) \mod (\psi - \theta_0)i^*\tilde{H}(E).$$

Then any ξ in $[\psi - \theta]_\pi(\alpha)$ can be modified by an element $\psi(x) - \theta x$ for some x in $\tilde{H}(E)$ so that the difference ξ_0 satisfies the requirements of the theorem.

4. **Indeterminacy for secondary operations.** In (2.1) we had a universal example $\Omega E_n \xrightarrow{i} E \xrightarrow{\pi} B$ and elements in $\tilde{H}(E)$ which were candidates for a universal secondary operation. If X is a finite complex and $f\colon X \to B$ a map representing a principal G-bundle with vanishing α-class, then we can lift f to a map from X to E, but the various liftings are not necessarily homotopic. We now calculate how these liftings differ on certain of the universal cohomology classes. We let $\theta(f)$ be the θ-class of the G-bundle determined by f, so that $f^*(\theta) = \theta(f)$. Choose ξ_0 as in (3.2).

(4.1) THEOREM. *Let f_1 and f_2 be maps from X to E such that πf_1 and πf_2 are both homotopic to f. Then*

$$f_1^*(\xi_0) = f_2^*(\xi_0) \mod (\psi-\theta(f))\tilde{H}^*(X).$$

Proof. Let $p: E\times \Omega E_n \to E$ be projection on the first factor. From [17, Lemma 1.2] we have a map $\mu: E\times\Omega E_n \to E$ so that $\pi\mu=\pi p$ and there is some map h from X to ΩE_n so that $\mu(f_1\times h)$ is homotopic to f_2. The functional operations $[\psi-\theta]_{\pi\mu}(\alpha)$ and $[\psi-\theta]_{\pi p}(\alpha)$ are defined and are equal. From (1.7)

$$[\psi-\theta]_{\pi\mu}(\alpha) = \mu^*[\psi-\theta]_\pi(\alpha) \mod (\psi-\mu^*\pi^*(\theta))\tilde{H}(E\times\Omega E_n)+\mu^*\pi^*\tilde{H}(B)$$

and

$$[\psi-\theta]_{\pi p}(\alpha) = p^*[\psi-\theta]_\pi(\alpha) \mod (\psi-p^*\pi^*(\theta))\tilde{H}(E\times\Omega E_n)+p^*\pi^*\tilde{H}(B).$$

From these relations we conclude that there is an x in $\tilde{H}(E_n\times\Omega E_n)$, b in $\tilde{H}(B)$ so that

$$\mu^*(\xi_0) = p^*(\xi_0)+\psi(x)-p^*\pi^*(\theta)x+p^*\pi^*(b). \tag{4.2}$$

Now we let $\eta: E\to E\times\Omega E_n$ be the map $\eta(e)=(e, c_0)$ where c_0 is the constant loop, and we notice that $p\eta=1$, $\mu\eta$ homotopic to 1. Apply η^* to (4.2) to get $\pi^*(b) = -(\psi(y)-\pi^*(\theta)y)$ where $y=\eta^*(x)$. Now we let $z=x-p^*(y)$ and we get

$$\mu^*(\xi_0) = p^*(\xi_0)+\psi(z)-p^*\pi^*(\theta)z.$$

Since $(f_1\times h)^*\mu^*=f_2^*$ and $p(f_1\times h)=f_1$ we find that

$$f_2^*(\xi_0) = (f_1\times h)^*\mu^*(\xi_1) = f_1^*(\xi_0)+\psi(w)-f_1^*\pi^*(\theta)w$$

where $w=(f_1\times h)^*(z)$. Since $f_1^*\pi^*(\theta)=\theta(f)$ we get the desired result. (*Note.* This is similar to Proposition 2.2 of [18]. Also compare this with Theorem 1 of [16].)

(4.3) With the data of (2.1) we can now define a secondary operation $\Phi(P, X)$ where P is a principal G-bundle over X with $\alpha(P)=0$. We let $f: X\to B$ represent P, and let $f_1: X\to E$ be a map such that $\pi f_1\sim f$. Then define $\Phi(P, X)=f_1^*(\xi_0)$ in $\tilde{H}^{n+N-1}(X)$ mod $(\psi-\theta(f))\tilde{H}^{N-1}(X)$. Clearly Φ is natural in P and X.

5. **Relation with the Thom isomorphism.** We have defined in (2.3) a secondary operation Ψ on G-bundles with vanishing α-class when there is a Thom isomorphism. The operation Φ (2.1) and (4.3) was defined universally. We want to relate these operations. We work with the universal fibering $\Omega E_n \xrightarrow{i} E \xrightarrow{\pi} B$ of (2.1) and the data of (2.3).

Let $Q\to B$ be the universal principal G-bundle over B and let $P\to E$ be the principal G-bundle induced over E by π. Then P is universal for principal G-bundles with vanishing α-class. Let $f: X\to B$ represent a principal G-bundle with vanishing α and $f_1: X\to E$ be a lifting to E. Then $f_1^{-1}(P)$ is the G-bundle represented by f. If u is the Thom class of $f_1^{-1}(P)$, $\theta(f)$ the θ-class of $f_1^{-1}(P)$, and (D, S) the disk and sphere bundles associated to $f_1^{-1}(P)$, $\eta: D\to D/S$ the zero section followed by projection onto the quotient, then it is easy to check that

$$\Psi(f_1^{-1}(P)) = [\psi-\theta(f)]_\eta(u).$$

Now, if $(D(P), S(P))$ is the Thom pair for P, then we have a commutative diagram

$$\begin{array}{ccc} X = D & \longrightarrow & D/S \\ f_1 \downarrow & & \downarrow f_1 \\ E = D(P) & \longrightarrow & D(P)/S(P) \end{array}$$

If we let U' be the Thom class of P so that $f_1^*(U') = u$, and $p\colon D(P) \to D(P)/S(P)$ the corresponding projection, then $[\psi - \theta(P)]_p(U')$ is defined, and by naturality of the functional operation,

$$f_1^*[\psi - \theta(P)]_p(U') = [\psi - \theta(f)]_\eta(u).$$

Thus a representative for $[\psi - \theta(P)]_p(U')$ is a universal example for the operation Ψ.

(5.1) THEOREM. *We can choose an element* ζ_0 *in* $\tilde{H}^{n+N-1}(E)$ *so that*

1. ζ_0 *represents* $[\psi - \theta]_\pi(\alpha)$.
2. ζ_0 *represents* $[\psi - \theta(P)]_p(U')$.
3. $i^*(\zeta_0) = -(\psi(\iota_{n-1}) - \theta_0 \iota_{n-1})$.

Now using ζ_0 as a universal example for both Ψ and Φ we have

(5.2) COROLLARY. *On any principal G-bundle over a finite complex with vanishing* α*-class*, $\Psi = -\Phi$.

REMARKS. One should compare this with Theorem 4.2 of [18]. We are not particularly confident that the minus sign is not a plus sign. Also, it has been pointed out to me by D. W. Anderson, that there is a Thom isomorphism in *KO*-theory for spin $(8n)$-bundles over infinite complexes, so that we might just as well hypothesize a Thom isomorphism for G-bundles over infinite complexes, and so we could speak of $\Psi(P)$ as well as $[\psi - \theta(P)]_\pi(U')$.

Proof of 5.1. We have a commutative diagram

$$\begin{array}{ccc} D(P)/S(P) & \xrightarrow{\pi'} & D(Q)/S(Q) \\ p \uparrow & & \uparrow q \\ E & \xrightarrow{\pi} & B \\ \uparrow & & \uparrow \\ S(P) & \longrightarrow & S(Q) \end{array}$$

We let U be the Thom class of P so that $\pi'^*(U) = U'$, $q^*(U) = \alpha$. By (1.8), (1.9) and (1.10) the functional operations $[\psi - \theta]_{q\pi}(U)$, $[\psi - \theta(P)]_\pi(\alpha)$, and $[\psi - \theta(P)]_p(U)$ are defined and

$$[\psi - \theta]_{q\pi}(U) = [\psi - \theta(P)]_\pi(\alpha) \mod \pi^*\tilde{H}(B) + (\psi - \theta(P))\tilde{H}(E),$$

$$[\psi - \theta]_{q\pi}(U) = [\psi - \theta(P)]_p(U') \mod p^*H(D(P), S(P)) + (\psi - \theta(P))\tilde{H}(E).$$

Let ζ represent $[\psi - \theta]_{q\pi}(U)$. Then by (3.1)

$$i^*(\zeta) = -(\psi(\iota_{n-1}) - \theta_0 \iota_{n-1}) + \psi(i^*(x)) - \theta_0 i^*(x).$$

Then let

$$\zeta_0 = \zeta-(\psi(x)-\theta(P)x)$$

so that ζ_0 still represents $[\psi-\theta]_{q\pi}(U)$ and $i^*(\zeta_0)=-(\psi(\iota_{n-1})-\theta_0\iota_{n-1})$. But ζ_0 also represents $[\psi-\theta(P)]_p(U')$.

6. **Secondary characteristic classes in *K*-theory.** If P is a principal spin$(2n)$-bundle over the finite complex X with $\Delta_{-1}(P)=0$, then corresponding to the relation $\psi^k(\Delta_{-1})=\theta_k\Delta_{-1}$ of (1.17) operations Φ_k and Ψ_k are defined on P via (2.1) and (2.3) and take values in $\tilde{K}U^{-1}(X)$ mod $(\psi^k-\theta_k(P))\tilde{K}U^{-1}(X)$. By 5 $\Phi_k(P)=-\Psi_k(P)$, and we want to compute these operations on some specific bundles.

Consider $U(2m+1)$ as a principal $U(2m)$-bundle over S_{4m+1} for $m>1$. Let E_m be the associated complex vector bundle. Then the sphere bundle is the complex Stiefel manifold $U(2m+1)/U(2m-1)$. $H^1(S_{4m+1})=H^2(S_{4m+1})=0$ so that E_m has a unique spin $(2m)$-reduction, P_m, and $\Delta_{-1}(P_m)=0$ since $\tilde{K}U^0(S_{4m+1})=0$. Thus $\Psi_k(P_m)$ is defined. From (1.24), $\tilde{K}U^{-1}(S_{4m+1})$ is infinite cyclic with generator γ^{2m+1}.

(6.1) THEOREM.

$$\Psi_k(P_m) = \pm\frac{(2m+1)}{2}k^{2m}(k-1)\gamma^{2m+1} \quad \text{mod}\ \{k^{2m}(k-1)\gamma^{2m+1}\}.$$

(6.2) COROLLARY. $\Psi_k(P_m)\neq 0$.

The proof will be given in §7.

If P is a principal spin $(8n)$-bundle over the finite complex X, then from (1.17), $\Psi^k(\Delta_{-1}(P))=\theta_k(P)\Delta_{-1}(P)$ in $\tilde{K}O^0(X)$, and if $\Delta_{-1}(P)=0$, then $\Psi_k(P)$ and $\Phi_k(P)$ are defined via (2.1) and (2.3) in $\tilde{K}O^{-1}(X)$ mod $(\Psi^k-\theta_k(P))\tilde{K}O^{-1}(X)$. Again $\Psi_k(P)=-\Phi_k(P)$ and we wish to compute this operation on some examples.

We consider $Sp(2m+1)$ as a principal $Sp(2m)$-bundle over S_{8m+3}, and we let F_m denote the associated real vector bundle. The sphere bundle of F_m is

$$Sp(2m+1)/Sp(2m-1).$$

As in the complex case, F_m has a unique spin $(8m)$ reduction Q_m, and $\Delta_{-1}(Q_m)=0$. Thus $\Psi_k(Q_m)$ is defined. Referring to the notation of (1.26) and (1.27) $\tilde{K}O^{-1}(S_{8m+3})$ is infinite cyclic generated by $f_m=\pi^{2m+1}\eta^{-1}\mu$.

(6.3) THEOREM.

$$\Psi_k(Q_m) = \pm\frac{(2m+1)}{24}k^{4m}(k^2-1)f_m \quad \text{mod}\ \{k^{4m}(k^2-1)f_m\}.$$

(6.4) COROLLARY. $\Psi_k(Q_m)\neq 0$.

The proof will be given in §8.

7. **Computations in the complex case.**

Proof of Theorem (6.1). Throughout this section K will denote KU. Let t_m be the Thom space of S_{4m+1} with respect to E_m. From (2.3) we know that the Gysin sequence splits up as

$$0\to\tilde{K}^{-1}(S_{4m+1})\to\tilde{K}^{-1}(U(2m+1)/U(2m-1))\to\tilde{K}^0(t_m)\to 0.$$

The group $\tilde{K}U^{-1}(U(2m+1)/U(2m-1))$ is $Z+Z$ with generators γ^{2m+1} and γ^{2m}, and so γ^{2m} projects onto $\pm$ the Thom class of E_m. Thus we must compute $\psi^k(\gamma^{2m})$. We know that

$$\psi^k(\gamma^{2m}) = \theta_k(P_m)\gamma^{2m}+b\gamma^{2m+1} \tag{7.1}$$

where $\pm b\gamma^{2m+1}$ represents $\Psi_k(P_m)$. Since $\tilde{K}^{-1}(U(2m+1)/U(2m-1))$ injects into $\tilde{K}^{-1}(U(2m+1))$, we can compute in $\tilde{K}^{-1}(U(2m+1))$. Further, we can compute in $RU(U(2m+1))\subset K^0(BU(2m+1))$ and apply the map

$$\delta^{-1}\pi^*\colon \tilde{K}^0(BU(2m+1)) \to K^0(EU(2m+1), U(2m+1)) \to \tilde{K}^{-1}(U(2m+1)). \tag{7.2}$$

This map has the advantage that it is zero on the square of the augmentation ideal. Finally, we can do the computation in $RU(T)$ where T is the diagonal torus in $U(2m+1)$.

We will refer to the notation of (1.18). We first define a virtual representation ρ^n of $U(n)$ by its character

$$\rho^n = \prod_{i=1}^{n}(1+\alpha_i+\cdots+\alpha_i^{k-1}).$$

Then

$$\psi^k(\gamma^n) = \prod_{i=1}^{n}(\alpha_i^k-1) = \gamma^n\rho^n.$$

Consider the restriction from $U(n)$ to $U(n-1)$. $\psi^k(\gamma^{n-1})$ restricts to $\psi^k(\gamma^{n-1})$, ρ^n restricts to $k\rho^{n-1}$, so $k\psi^k(\gamma^{n-1})-\gamma^{n-1}\rho^n$ is in the kernel of the restriction. Thus

$$k\psi^k(\gamma^{n-1})-\gamma^{n-1}\rho^n = a\gamma^n+\gamma^nP(\gamma^1,\ldots,\gamma^n) \tag{7.3}$$

where a is some integer and P is a polynomial with no constant term. If we write these representations in terms of characters and divide by γ^n we get

$$\sum_{i=1}^{n}\frac{1}{\alpha_i-1}\left\{k\prod_{j\neq i}(1+\alpha_j+\cdots+\alpha_j^{k-1})-\prod_{i=1}^{n}(1+\alpha_i+\cdots+\alpha_i^{k-1})\right\} = a+P(\gamma^1,\ldots\gamma^n).$$

Now, we factor the expression in brackets, and letting

$$s_i = \prod_{j\neq i}(1+\alpha_j+\cdots+\alpha_j^{k-1}),$$

$$t_i = 1+(1+\alpha_i)+\cdots+(1+\alpha_i+\cdots+\alpha_i^{k-2}),$$

we get

$$-\sum_{i=1}^{n}s_it_i = a+P(\gamma^1,\ldots,\gamma^n).$$

Then we restrict to the trivial subgroup of $U(n)$, where $\gamma^i=0$, $\alpha_i=1$ and find that $a=-nk^n(k-1)/2$.

Next we write $\rho^n = c + R(\gamma^1, \ldots, \gamma^n)$, where c is an integer and R is a polynomial with no constant term. We let $\gamma^1 = \cdots = \gamma^n = 0$ and find that $c = k^n$. Thus (7.3) becomes

$$k\psi^k(\gamma^{n-1}) = k^n\gamma^{n-1} - \frac{nk^n(k-1)}{2}\gamma^n + \gamma^n P + \gamma^{n-1}R. \tag{7.4}$$

Now we let $n = 2m+1$ and apply the map of (7.2) to get

$$\psi^k(\gamma^{2m}) = k^{2m}\gamma^{2m} - \frac{(2m+1)}{2}k^{2m}(k-1)\gamma^{2m+1}. \tag{7.5}$$

Thus

$$\theta_k(P_m) = k^{2m} \quad \text{and} \quad b = -\frac{(2m+1)}{2}k^{2m}(k-1).$$

The indeterminacy of $\Psi_k(P_m)$ is generated by $\psi^k(\gamma^{2m+1}) - k^{2m}\gamma^{2m+1}$. In $RU(U(2m+1))$

$$\psi^k(\gamma^{2m+1}) = \gamma^{2m+1}\rho^{2m+1} = k^{2m+1}\gamma^{2m+1} + \gamma^{2m+1}R(\gamma^1, \ldots, \gamma^{2m+1}).$$

Thus $\psi^k(\gamma^{2m+1}) = k^{2m+1}\gamma^{2m+1}$ in $\tilde{K}^{-1}(U(2m+1))$ and so the indeterminacy is generated by $k^{2m}(k-1)\gamma^{2m+1}$.

8. **Computations in the real case.**

Proof of Theorem (6.3). We let t_m denote the Thom space of S_{8m+3} with respect to F_m. Then the Gysin sequence for F_m in KO-theory gives rise to a short exact sequence

$$0 \to \widetilde{KO}^{-1}(S_{8m+1}) \to \widetilde{KO}^{-1}(Sp(2m+1)/Sp(2m-1)) \to \widetilde{KO}^0(t_m) \to 0.$$

The middle term is $Z+Z$ with generators f_m and π^{2m}. Thus by (2.3)

$$\psi^k(\pi^{2m}) = \theta_k(Q_m)\pi^{2m} + bf_m, \tag{8.1}$$

where b is an integer and $\pm bf_m$ represents $\Psi_k(Q_m)$. From (1.26) and (1.27) the map

$$\varepsilon_U\colon \widetilde{KO}^{-1}(Sp(2m+1)/Sp(2m-1)) \to \widetilde{KU}^{-1}(Sp(2m+1)/Sp(2m-1))$$

takes π^{2m} to π^{2m} and f_m to $2\pi^{2m+1}$ so that ε_U is injective and we need only compute

$$\psi^k(\pi^{2m}) = \theta_k(Q_m)\pi^{2m} + (2b)\pi^{2m+1} \tag{8.2}$$

in $\tilde{K}^{-1}(Sp(2m+1)/Sp(2m-1))$. By (1.26) we can do this computation in $\tilde{K}^{-1}(Sp(2m+1))$, and as in the complex case, we can compute in $RU(Sp(2m+1))$ and apply the map

$$\delta^{-1}\pi^*\colon \tilde{K}^0(BSp(2m+1)) \to \tilde{K}^{-1}(Sp(2m+1)). \tag{8.3}$$

Again, we will do the computation in $RU(T)$ where T is a maximal torus of $Sp(2m+1)$.

Now we work with $Sp(n)$ and refer to (1.19). We define the virtual representation ρ^n of $Sp(n)$ by its character

$$\rho^n = \prod_{i=1}^{n} (1+\alpha_i+\cdots+\alpha_i^{k-1})(1+\alpha_i^{-1}+\cdots+\alpha_i^{-(k-1)}).$$

Then it follows that $\psi^k(\pi^n)=\rho^n\pi^n$. Under restriction from $Sp(n)$ to $Sp(n-1)$, ρ^n restricts to $k^2\rho^{n-1}$ so $k^2\psi^k(\pi^{n-1})-\rho^n\pi^{n-1}$ is in the kernel of the restriction map. Thus

$$k^2\psi^k(\pi^{n-1})-\rho^n\pi^{n-1} = a\pi^n+\pi^nP(\pi^1,\ldots,\pi^n) \tag{8.4}$$

where a is an integer and P is a polynomial with no constant term. Now

$$\psi^k(\pi^{n-1}) = \sum_{i=1}^{n}\left\{\prod_{j\neq i}(\alpha_j^k+\alpha_j^{-k}-2)\right\}$$

and

$$\alpha^k+\alpha^{-k}-2 = (\alpha+\alpha^{-1}-2)(1+\alpha+\cdots+\alpha^{k-1})(1+\alpha^{-1}+\cdots+\alpha^{-(k-1)}).$$

Using this formula and dividing by π^n (8.4) becomes

$$\sum_{i=1}^{n}\left\{\frac{t_i}{\alpha_i+\alpha_i^{-1}-2}(k^2-(1+\alpha_i+\cdots+\alpha_i^{k-1})(1+\alpha_i^{-1}+\cdots+\alpha_i^{-(k-1)}))\right\}$$
$$= a+P(\pi^1,\ldots,\pi^n)$$

where

$$t_i = \prod_{j\neq i}(1+\alpha_j+\cdots+\alpha_j^{k-1})(1+\alpha_j^{-1}+\cdots+\alpha_j^{-(k-1)}).$$

Now we write

$$k^2-(1+\alpha_i+\cdots+\alpha_i^{k-1})(1+\alpha_i^{-1}+\cdots+\alpha_i^{-(k-1)}) = -(\alpha_i+\alpha_i^{-1}-2)s_i$$

where

$$s_i = (1+\alpha_i+\cdots+\alpha_i^{k-2})(1+\alpha_i^{-1}+\cdots+\alpha_i^{-(k-2)})$$
$$+2(1+\alpha_i+\cdots+\alpha_i^{k-3})(1+\alpha_i^{-1}+\cdots+\alpha_i^{-(k-3)})+\cdots+(k-1).$$

Then (8.4) becomes

$$-\sum_{i=1}^{n} s_it_i = a+P(\pi^1,\ldots,\pi^n).$$

Now to find a, we set $\pi^1=\cdots=\pi^n=0$, $\alpha_j=1$ and use the fact that

$$\dim s_i = \sum_{j=1}^{k-1} j^2(k-j) = k^2(k-1)(k+1)/12.$$

We find that

$$a = -\frac{nk^{2(n-1)}k^2(k^2-1)}{12}.$$

As in the complex case we write $\rho^n=k^{2n}+R(\pi^1,\ldots,\pi^n)$ where R is a polynomial with no constant term. We apply $\delta^{-1}\pi^*$ of (8.3) to (8.4) and let $n=2m+1$ to obtain

$$\psi^k(\pi^{2m}) = k^{4m}\pi^{2m}-\frac{(2m+1)k^{4m}(k^2-1)}{12}\pi^{2m+1} \tag{8.5}$$

in $\tilde{K}^{-1}(Sp(2m+1)/Sp(2m-1))$. Thus in $\tilde{KO}^{-1}(Sp(2m+1)/Sp(2m-1))$ we have

$$\psi^k(\pi^{2m}) = k^{4m}\pi^{2m} - \frac{(2m+1)k^{4m}(k^2-1)}{24} f_m.$$

So, $\pm(2m+1)k^{4m}(k^2-1)f_m/24$ represents $\Psi_k(Q_m)$ in

$$\tilde{KO}^{-1}(S_{8m+3}) \quad \text{mod } \{\psi^k(f_m) - k^{4m}f_m\}.$$

To find $\psi^k(f_m)$ it is enough to compute $\psi^k(\pi^{2m+1})$ in $\tilde{K}^{-1}(S_{8m+3})$. In $RU(Sp(2m+1))$ we know that

$$\psi^k(\pi^{2m+1}) = \rho^{2m+1}\pi^{2m+1} = k^{4m+2}\pi^{2m+1} + \pi^{2m+1}R(\pi^1, \ldots, \pi^{2m+1}).$$

Thus $\psi^k(f_m) = k^{4m+2}f_m$ in $\tilde{KO}^{-1}(S_{8m+3})$ and the indeterminacy of $\Psi_k(Q_m)$ is generated by $k^{4m}(k-1)f_m$.

BIBLIOGRAPHY

1. D. W. Anderson, *Universal coefficient theorems for K-theory*, (to appear).

2. D. W. Anderson, E. H. Brown, Jr. and F. P. Peterson, SU-*cobordism*, KO-*characteristic numbers, and the Kervaire invariant*, Ann. of Math. **83** (1966), 54–67.

3. M. F. Atiyah, *Immersions and embeddings of manifolds*, Topology **1** (1962), 125–133.

4. ———, *Lectures on K-theory*, mimeographed notes, Harvard Univ., Cambridge, Mass., 1964.

5. M. F. Atiyah, R. Bott and A. Shapiro, *Clifford modules*, Topology **3** (Suppl. 1) (1964), 3–38.

6. M. F. Atiyah and F. Hirzebruch, "Vector bundles and homogeneous spaces" in *Differential geometry*, Proc. Sympos. Pure Math., Vol. 3, Amer. Math. Soc., Providence, R. I., 1961.

7. R. Bott, *Lectures on K(X)*, mimeographed notes, Harvard Univ., Cambridge, Mass., 1963.

8. A. Dold, *Relations between ordinary and extraordinary homology*, Colloq. on Algebraic Topology, Aarhus Univ., Aarhus, 1962, pp. 2–9.

9. B. Eckmann and P. J. Hilton, *Operators and cooperators in homotopy theory*, Math. Ann. **141** (1960), 1–21.

10. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, Princeton, N. J., 1952.

11. W. S. Massey, *On the embeddability of real projective spaces in Euclidean space*, Pacific J. Math. **9** (1959), 783–789.

12. J. Milnor, *On axiomatic homology theory*, Pacific J. Math. **12** (1962), 337–341.

13. ———, *On spaces having the homotopy type of a* CW-*complex*, Trans. Amer. Math. Soc. **90** (1959), 272–280.

14. ———, *Representation rings of some classical groups*, mimeographed notes, Princeton Univ., Princeton, N. J., 1963.

15. F. P. Peterson, *Functional cohomology operations*, Trans. Amer. Math. Soc. **86** (1957), 197–211.

16. ———, *Principal fibre spaces*, Amer. J. Math. **82** (1960), 649–652.

17. F. Peterson and N. Stein, *Secondary cohomology operations*, Amer. J. Math. **81** (1959), 281–305.

18. ———, *Secondary characteristic classes*, Ann. of Math. **76** (1962), 510–523.

19. D. Puppe, *Homotopiemengen und ihre induzierten Abbildungen*, Math. Z. **69** (1958), 299–344.

20. Seminaire Henri Cartan, *Périodicité des groupes d'homotopie stables des groupes classiques*, École Norm. Sup., Paris, 1959/1960.

21. J. H. C. Whitehead, *Combinatorial homotopy*, Bull. Amer. Math. Soc. **55** (1949), 213–245.

22. G. W. Whitehead, *Generalized homology theories*, Trans. Amer. Math. Soc. **102** (1962), 227–282.

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